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The Super-Halley Method Using Divided Differences

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Abstract—Semilocal convergence results are provided for the super-Halley method in a Banach space at the first part of this study. In the second part, the monotone convergence is examined in a partially ordered topological setting.

Keywords—Super-Halley method, Banach space, POTL-space, Divided differences.

1. INTRODUCTION

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear operator equation

$$F(x) = 0. \tag{1}$$

In Section 1, we assume that the operator F is defined on some convex subset D of a Banach space B_1 with values in a Banach space B_2. In Section 2, we assume that F is defined on a convex subset D of partially ordered topological space (POTL-space) B_1 with values in a POTL-space B_2.

The method of tangent hyperbolas or the Chebysheff-Halley method is a cubically convergent procedure for solving equation (1). This method has been generalized for nonlinear operator equations in Banach space [1–5]. It is given by

$$z_{n+1} = z_n - \left\{ I - \frac{1}{2} F'(z_n)^{-1} F''(z_n) F'(z_n)^{-1} F(z_n) \right\}^{-1} F'(z_n)^{-1} F(z_n), \qquad n \geq 0, \tag{2}$$

for some $z_0 \in D$. With the exception of some special cases, these generalizations have little practical value because they require an evaluation of the second Fréchet derivative at each step (which means a number of function evaluations proportional to the cube of the dimension of the space). Discretized versions of method (2) have already been considered using divided differences of order two, which also makes the method unattractive from the numerical point of view.

Here, we introduce the super-Halley methods defined for all $n \geq 0$ by

$$y_n = x_n - [x_{n-1}, x_n]^{-1} F(x_n), \tag{3}$$

$$H_n = 2 [x_{n-1}, x_n]^{-1} ([x_{n-1}, y_n] - [x_{n-1}, x_n]), \tag{4}$$

$$x_{n+1} = y_n - \frac{1}{2} \left[I + \frac{\lambda}{2} H_n \right]^{-1} H_n (y_n - x_n) \tag{5}$$

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for the Banach space case, and

$$F(v_n) + [x_{n-1}, x_n](w_n - v_n) = 0, \tag{6}$$

$$F(x_n) + [x_{n-1}, x_n](y_n - x_n) = 0, \tag{7}$$

$$([x_{n-1}, y_n] - [x_{n-1}, x_n])(w_n - v_n) + \left[\left(1 - \frac{\lambda}{2}\right)[x_{n-1}, x_n] + \frac{\lambda}{2}[x_{n-1}, y_n]\right] \times (v_{n+1} - w_n) = 0, \tag{8}$$

$$([x_{n-1}, y_n] - [x_{n-1}, x_n])(y_n - x_n) + \left[\left(1 - \frac{\lambda}{2}\right)[x_{n-1}, x_n] + \frac{\lambda}{2}[x_{n-1}, y_n]\right] \times (x_{n+1} - y_n) = 0, \tag{9}$$

for the POTL-space case, where x_{-1}, x_0, v_0 are given fixed points in D. By $[x, y]$, we denote a divided difference of order one on D satisfying

$$[x, y](x - y) = F(x) - F(y) \tag{10}$$

for all $x, y \in D$ (see [1,3,5]).

Note that our methods involve only divided differences of order one, which makes them attractive from the numerical point of view.

We provide semilocal convergence results for the iteration $\{x_n\}$ $(n \geq -1)$ generated by (3)–(5) in a Banach space as well as monotone convergence results for the iterations $\{x_n\}$, $\{v_n\}$ $(n \geq -1)$ generated by (6)–(9) in a POTL-space.

2. SEMILOCAL CONVERGENCE ANALYSIS

Let $x_{-1} \neq x_0 \in D$ be fixed. It is convenient to introduce the parameters λ, t_{-1}, t_0, s_0, d, c by

$$\lambda \in \mathbf{R}, \qquad t_{-1} \geq 0,$$
$$\|x_{-1} - x_0\| \leq t_0 - t_{-1} \leq d,$$
$$\|y_0 - x_0\| \leq s_0 - t_0,$$
$$q = \frac{1 - cd}{2(1 + 2|\lambda|)c},$$

the sequences and functions

$$\begin{aligned}
\bar{a}_n &= \left[1 - c\left(\|x_{n-1} - x_0\| + \|x_n - x_0\| + \|x_{-1} - x_0\|\right)\right]^{-1}, \\
a_n &= \left[1 - c\left(t_n + t_{n-1} + d\right)\right]^{-1}, \\
a(r) &= [1 - c(2r + d)]^{-1}, \\
\bar{b}_n &= |\lambda| \bar{a}_n c \left(2\|x_{n-1} - x_0\| + \|x_n - x_0\| + \|y_n - x_0\|\right), \\
b_n &= |\lambda| a_n c \left(2t_{n-1} + t_n + s_n\right), \\
b(r) &= \frac{4|\lambda| cr}{1 - c(2r + d)}, \\
\bar{d}_n &= \left(1 - \bar{b}_n\right)^{-1}, \\
d_n &= \left(1 - b_n\right)^{-1}, \\
d(r) &= (1 - b(r))^{-1}, \\
\bar{e}_{n+1} &= a_{n+1}\left[\left(\|x_{n+1} - y_n\| + 2\|y_n - x_n\| + \|x_n - x_{n-1}\|\right)\|x_{n+1} - y_n\|\right. \\
&\quad \left. + \left(\left(1 + 2\bar{d}_n\right)\|x_{n-1} - x_n\| + \left(1 + \bar{d}_n\right)\|y_n - x_n\|\right)\|y_n - x_n\|\right],
\end{aligned}$$

$$
\begin{aligned}
e_{n+1} &= a_{n+1}\left(t_{n+1}+s_n-t_n-t_{n-1}\right)\left(t_{n+1}-s_n\right) \\
&\quad + \left(s_n-t_{n-1}+d_n\left(t_n+s_n-2t_{n-1}\right)\right)\left(s_n-t_n\right), \\
e(r) &= a(r)(3+2d(r))r^2, \\
\bar{p}_n &= \left[1-c\left(\|x^*-x_0\|+\|x_n-x_0\|+\|x_{-1}-x_0\|\right)\right]^{-1}, \\
T(r) &= s_0+e(r)+2d(r)r^2, \\
t_{n+1} &= s_n+d_n\left(t_n+s_n-2t_{n-1}\right)\left(s_n-t_n\right)
\end{aligned}
\tag{11}
$$

and

$$
s_{n+1} = t_{n+1}+e_n, \qquad n \geq 0. \tag{12}
$$

Given $x_{-1} \neq x_0 \in D$, the parameters t_{-1}, t_0, s_0, and d are used to measure the initial distances $\|x_{-1}-x_0\|$ and $\|y_0-x_0\|$ (via (3)–(5)). The sequences $\bar{a}_n$, $\bar{b}_n$, $\bar{d}_n$, $\bar{e}_n$, $\bar{p}_n$, $(n \geq 0)$ are used to find upper bounds on the distances $\|x_{n+1}-y_n\|$ and $\|x_n-y_n\|$, $(n \geq 0)$. The real functions a, b, d, and e defined on $[0, r_0]$ (some $r_0 \geq 0$) are used as upper bounds on the sequences a_n, b_n, d_n, and e_n, $(n \geq 0)$, which in turn are upper bounds on the sequences $\bar{a}_n$, $\bar{b}_n$, $\bar{d}_n$, and $\bar{e}_n$, respectively, for all $n \geq 0$. Finally, function T defined on $[0, r_0]$ is such that $t_n \leq s_n \leq t_{n+1} \leq s_{n+1} \leq T(r_0) \leq r_0$, $(n \geq 0)$.

With the notation introduced above, we can now state the main result of this section. The proof will appear elsewhere (however, see [1, Chapter 5]).

THEOREM 1. *Let F be a Fréchet-differentiable nonlinear operator defined on a convex subset D of a Banach space B_1 with values in a Banach space B_2.*

Assume:

(a) *there are points $x_{-1} \neq x_0$ with $x_{-1}, x_0 \in D$, such that the linear operator $A_0 = [x_{-1}, x_0]$ (or $[x_0, x_{-1}]$) is invertible;*

(b) *the nonlinear operator F has divided differences of order one satisfying the Lipschitz condition*

$$
\left\|A_0^{-1}\left([x,y]-F'(z)\right)\right\| \leq c(\|x-z\|+\|y-z\|)
$$

for all $x, y, z \in D$, and some $c \geq 0$;

(c) *there exists a minimum positive number r_0 satisfying the inequality*

$$
T(r_0) \leq r_0;
$$

(d) *the point r_0 also satisfies*

$$
r_0 < q
$$

provided that

$$
cd < 1;
$$

and

(e) *the following inclusion is true:*

$$
U(x_0, r_0) = \{x \in B_1 \mid \|x-x_0\| \leq r_0\} \subseteq D.
$$

Then,

(i) *the scalar iterations $\{t_n\}$, $(n \geq -1)$ and $\{s_n\}$, $(n \geq 0)$ generated by (11) and (12), respectively, are monotonically increasing with $0 \leq t_{n-1} \leq s_n \leq t_n \leq s_{n+1}$ for all $n \geq 0$, and bounded above by their common limit r_0.*

(ii) *The iteration* $\{x_n\}$, $(n \geq -1)$ *generated by (3)–(5) is well defined, remains in* $U(x_0, r_0)$ *for all* $n \geq 0$, *and converges to some* $x^* \in U(x_0, r_0)$, *which is the unique solution of the equation* $F(x) = 0$ *in* $U(x_0, r_0)$. *Moreover, the following estimates are true:*

$$\begin{aligned} \|x_{n+1} - y_n\| &\leq t_{n+1} - s_n, \\ \|y_{n+1} - x_{n+1}\| &\leq s_{n+1} - t_{n+1}, \\ \|x_n - x^*\| &\leq r_0 - t_n, \\ \|y_n - x^*\| &\leq r_0 - s_n \end{aligned}$$

and

$$\|x_n - x^*\| \leq \frac{\bar{p}_n \bar{e}_n}{\bar{a}_n}, \qquad n \geq 0.$$

3. MONOTONE CONVERGENCE

It is convenient to define the following divided differences

$$A_n = [x_{n-1}, x_n], \qquad L_n = [x_{n-1}, y_n], \qquad R_n = \left(1 - \frac{\lambda}{2}\right) A_n + \frac{\lambda}{2} L_n,$$

and M_n, $\bar{R}_n$ denote the continuous left subinverses of A_n and R_n, respectively, for all $n \geq 0$.

We can now state the theorem whose proof will appear elsewhere (see also [1, Chapter 6]).

THEOREM 2. *Let* F *be a nonlinear operator defined on a convex subset* D *of a regular POTL-space* B_1 *with values in a POTL-space* B_2. *Let* v_0, x_0, *and* x_{-1} *be three points of* D *such that*

$$v_0 \leq x_0 \leq x_{-1}$$

and

$$F(v_0) \leq 0 \leq F(x_0).$$

Suppose that F *has a divided difference of order one on* $D_0 = \langle v_0, x_{-1}\rangle = \{x \in B_1 \mid v_0 \leq x \leq x_{-1}\} \subseteq D$ *satisfying:* A_0 *has a continuous nonnegative left subinverse* M_0,

$$\begin{aligned} [x_{-1}, y] &\geq 0, && \text{for all } v_0 \leq y \leq x_{-1}, && (13) \\ [x, y] &\leq [z, w], && \text{if } x \leq z \text{ and } y \leq w, && (14) \\ (A_n - L_n) M_n [x_n, v_n] &\leq R_n, && \text{for all } n \geq 0, && (15) \\ M_n [v_n, x_n] + \bar{R}_n (A_n - L_n) M_n [v_n, x_n] &\leq I, && n \geq 0, && \end{aligned}$$

and

$$R_n \geq 0, \qquad \text{for all } n \geq 0. \tag{16}$$

Then there exist two sequences $\{v_n\}$, $\{x_n\}$, $(n \geq 0)$ *satisfying the approximations (6)–(9),*

$$\begin{aligned} &v_0 \leq w_0 \leq v_1 \leq \cdots \leq w_n \leq v_{n+1} \leq x_{n+1} \leq y_n \leq \cdots \leq x_1 \leq y_0 \leq x_0, \\ &\lim_{n\to\infty} v_n = v^*, \qquad \lim_{n\to\infty} x_n = x^*, \qquad \text{and} \qquad v^*, x^* \in D \quad \text{with} \quad v^* \leq x^*. \end{aligned}$$

Moreover, if the linear operators A_n *are inverse nonnegative, then any solution* u *of the equation* $F(x) = 0$ *in* $\langle v_0, x_{-1}\rangle$ *belongs to* $\langle v^*, x^*\rangle$.

REMARKS.

(a) Condition (16) can certainly be replaced by $\lambda \in [0, 2]$.

(b) Condition (15) can be replaced by the hypothesis that M_n is just a subinverse of A_n for all $n \geq 0$. Note then, that (15) reduces to showing that $y_n \leq x_n$, which is true for all $n \geq 0$.

(c) Condition (13) can also be replaced by

$$y_n + \frac{\lambda}{2}(y_n - x_n) \geq 0, \qquad n \geq 0$$

or

$$y_n + \frac{\lambda}{2}(y_n - x_n) \in \langle v_0, x_{-1} \rangle, \qquad n \geq 0.$$

(d) If we set $\lambda = 0$, and replace conditions (15), (16) with the ones mentioned in (a), (b) above, then our theorem shows convergence with conditions simpler to verify than the ones obtained in [3] for a similar algorithm. In particular, see [3, Condition (50)].

(e) Conditions of the form (13) and (14) appear frequently in the study of monotone convergence of Newton-like methods. In the case $B_1 = B_2 = \mathbf{R}$, these conditions are satisfied if and only if F is differentiable on D_0 and F, F' are convex on D_0 (see [1]).

(f) Conditions for $x^* = y^*$ can be found in [1, Chapter 6].

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